Negative moments of characteristic polynomials of random GOE matrices and singularity-dominated strong fluctuations

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Abstract
We calculate the negative integer moments of the (regularized) characteristic polynomials of $N \times N$ random matrices taken from the Gaussian orthogonal ensemble (GOE) in the limit as $N \to \infty$. The results agree nontrivially with a recent conjecture of Berry and Keating motivated by techniques developed in the theory of singularity-dominated strong fluctuations. This is the first example where nontrivial predictions obtained using these techniques have been proved.

1. Introduction

Let $\hat{H} = \hat{H}^T$ (we here use the symbol T to denote matrix or vector transposition and $*$ to denote complex conjugation) be an $N \times N$ random symmetric matrix with real entries distributed according to the standard joint probability density of the Gaussian orthogonal ensemble (GOE) of random matrix theory,

$$\mathcal{P}(\hat{H}) = C_N \, \mathrm{e}^{-\frac{N}{2J^2}\mathrm{Tr}\hat{H}^2} \tag{1}$$

with respect to the measure $\mathrm{d}\hat{H} = \prod_{i=1}^N \mathrm{d}H_{ii} \prod_{i<j} \mathrm{d}H_{ij}$, where the normalization constant C_N is given by

$$C_N = \frac{1}{2^{N/2}} \left(\frac{N}{\pi J^2} \right)^{\frac{N(N+1)}{4}} \tag{2}$$

and let

$$Z_N(\mu) = \det(\mu \mathbf{1}_N - \hat{H}) \tag{3}$$

denote its characteristic polynomial. We here are interested in the negative integer moments of the absolute value $|Z|$, defined by averaging over the GOE, when $\mathrm{Im}\,\mu > 0$, in the limit

as $N \to \infty$. (The positive moments of the characteristic polynomials of random unitary-symmetric matrices were calculated in [14]; for the positive integer moments it was confirmed in [5] that, as expected, these results also apply to the large N limit of matrices in the GOE; see also [16].)

Berry and Keating [3] (hereinafter referred to as BK) have recently put forward a general conjecture about the asymptotics of the negative moments of the absolute values of the characteristic polynomials[3] of random matrices in the limit as the matrix size tends to infinity when Im μ is scaled by the mean eigenvalue density and tends to zero. This conjecture applies to all negative moments, rather than just to negative integer moments, and covers all the three classical random matrix ensembles (i.e. the unitary, orthogonal and symplectic ensembles). It predicts a highly nontrivial dependence of the asymptotics on the power to which the polynomial is raised. This is in contrast to the case when the large-matrix limit is taken without scaling Im μ by the mean level spacing; then the moment asymptotics is much simpler [11].

In the case of the Gaussian unitary ensemble (GUE) of random matrices, the conjecture given in BK agrees with the values of the negative integer moments calculated by Fyodorov in [6] and shown to be universal (in the sense that they apply to all unitary-invariant ensembles of Hermitian matrices) in [19]. However, these values also happen to coincide with the corresponding ones when Im μ is not scaled, so this cannot be said to constitute a test of the nontrivial aspects of the conjecture.

For the GOE of random matrices the conjecture in BK is that the ensemble average of $|Z_N(\mu)|^{-k}$ diverges like $\epsilon^{-\nu(k)}$, as ϵ, Im μ scaled by the mean eigenvalue density, tends to zero, with

$$\nu(k) = \text{int}(k) \left(k - \frac{1 + \text{int}(k)}{2} \right). \tag{4}$$

It was suggested in BK (page L4) that, in the notation of the present paper, when k is an integer it is possible that the leading-order power-law behaviour (4) is multiplied by a power of $\log \frac{1}{\epsilon}$.

Our first aim here is to extend the heuristic arguments developed in BK to recover the logarithmic factor when k takes integral values; this turns out to be simply $\log \frac{1}{\epsilon}$ for each k. Our second aim then is to prove the resulting expression by a direct evaluation of the GOE average. In fact, we are able to go significantly further in that we calculate the precise asymptotic form of the moments in the appropriate limit. The general expression we obtain (see (40) and (41)) takes the form of a multiple integral and is interesting in its own right, in particular in view of recent endeavours to understand the analytic structure behind the so-called replica limit $k \to 0$ [12, 18]. An asymptotic evaluation of this integral leads to our final result, given in equation (52).

The heuristic arguments described in BK, which motivate the conjecture made there, are an application of general techniques associated with the theory of *singularity-dominated strong fluctuations*. These techniques have been previously applied to analyse twinkling starlight [1], van Hove-type singularities [2] and the influence of classical periodic orbit bifurcations on quantum energy level [4] and wavefunction [13] statistics. In all these applications the results correspond to power-law asymptotics of the moments of fluctuating quantities as the relevant parameter vanishes, with exponents that emerge from a competition between different singular contributions. It has been shown for the moments of the intensity fluctuations beyond a one-dimensional refracting screen that exactly when one kind of singularity overtakes another in

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[3] Note that if the absolute value is not taken, the negative moments are non-singular, see, e.g., [6]. In this case a thorough study of the moments, in the framework of the theory of hypergeometric functions in several variables, can be found in [22].

the competition there is an additional logarithmic factor [8, 9]. (The constants multiplying the various asymptotic contributions were also obtained in this case.) Importantly, in none of the applications studied previously has it been possible to prove nontrivial predictions of the theory of singularity-dominated strong fluctuations by an asymptotic analysis that could be made rigorous.

In the example we study here, the singularity competition considered in BK is between clusters of nearly degenerate eigenvalues. Clusters involving p eigenvalues give rise to a contribution to the ensemble average of $|Z_N(\mu)|^{-k}$ that diverges like $\epsilon^{-\nu_p(k)}$ as $\epsilon \to 0$. For a given k, the dominating cluster size is the one for which the exponent $\nu_p(k)$ is maximal. It was shown in BK that this produces the exponent (4). Here, in section 2, we show that for k an integer, when one p takes over from another as dominant, there is an additional logarithmic factor, as described above. In section 3, we prove this result by calculating the GOE average explicitly, in the large matrix-size limit. This represents the first example where nontrivial predictions of theory of singularity-dominated strong fluctuations have been proved.

2. Cluster contributions

We here re-analyse the arguments presented in BK to recover explicitly the logarithmic factor anticipated there in the case of negative integer moments of characteristic polynomials of random matrices in the GOE.

Let $M_p(-k, \epsilon)$ denote the contribution from clusters of p eigenvalues (we henceforth refer to this as the *p-cluster* contribution) to the GOE average of $|Z_N(\mu)|^{-k}$, where ϵ is $\text{Im}\,\mu$ scaled by the mean eigenvalue density. Treating a p-cluster of unfolded levels $x_1, x_2, \ldots, x_p$ as isolated from the rest of the spectrum, the contribution to the average, $\prod_{n=1}^{p} \left(x_n^2 + \epsilon^2\right)^{-k/2}$, and the joint probability density function, $P(x_1, x_2, \ldots, x_p) \propto \prod_{m=1}^{p-1} \prod_{n=m+1}^{p} |x_m - x_n|$, combine to give (cf equations (9) and (10) of BK)

$$M_p(-k, \epsilon) \propto \int_{-X}^{X} dx_1 \int_{-X}^{X} dx_2 \cdots \int_{-X}^{X} dx_p \frac{\prod_{m=1}^{p-1} \prod_{n=m+1}^{p} |x_m - x_n|}{\left[\left(x_1^2 + \epsilon^2\right)\left(x_2^2 + \epsilon^2\right) \cdots \left(x_p^2 + \epsilon^2\right)\right]^{k/2}}. \tag{5}$$

To be precise, the limits of integration were given as $-\infty$ and ∞ in BK. This distinction will be important when k is an integer, and not otherwise. A finite integration range is, in fact, more appropriate; in the case of the circular ensembles of random matrix theory because the eigenphases lie in a finite interval, and in the case of the Gaussian (or similar) ensembles because the potential effectively limits the range in which the eigenvalues lie.

Making the change of variables $x_m = \epsilon u_m$ gives

$$M_p(-k, \epsilon) \propto \epsilon^{\frac{p(p+1)}{2} - pk} \int_{-X/\epsilon}^{X/\epsilon} du_1 \int_{-X/\epsilon}^{X/\epsilon} du_2 \cdots$$
$$\times \int_{-X/\epsilon}^{X/\epsilon} du_p \frac{\prod_{m=1}^{p-1} \prod_{n=m+1}^{p} |u_m - u_n|}{\left[\left(u_1^2 + 1\right)\left(u_2^2 + 1\right) \cdots \left(u_p^2 + 1\right)\right]^{k/2}}. \tag{6}$$

It was demonstrated in BK that the p-cluster contribution dominates the kth moment when $p \leqslant k < p + 1$. It is straightforward to see that the integral in (6) converges as $\epsilon \to 0$ in the range $p < k < p + 1$. It is then asymptotically consistent to replace the limits of integration by $-\infty$ and ∞, and the results of BK hold without change. When k is an integer, the $p = k$ integral diverges and so must be treated more carefully.

Let

$$I_p(X/\epsilon) = \int_{-X/\epsilon}^{X/\epsilon} du_1 \int_{-X/\epsilon}^{X/\epsilon} du_2 \cdots \int_{-X/\epsilon}^{X/\epsilon} du_p \frac{\prod_{m=1}^{p-1} \prod_{n=m+1}^{p} |u_m - u_n|}{\left[\left(u_1^2 + 1\right)\left(u_2^2 + 1\right) \cdots \left(u_p^2 + 1\right)\right]^{p/2}}. \tag{7}$$

Consider first the case when $p = 1$:

$$I_1(X/\epsilon) = \int_{-X/\epsilon}^{X/\epsilon} \frac{du_1}{\left(u_1^2 + 1\right)^{1/2}} \tag{8}$$

which clearly diverges like $\log \frac{1}{\epsilon}$ as $\epsilon \to 0$.

Consider next the case when $p = 2$:

$$I_2(X/\epsilon) = \int_{-X/\epsilon}^{X/\epsilon} du_1 \int_{-X/\epsilon}^{X/\epsilon} du_2 \frac{|u_1 - u_2|}{\left(u_1^2 + 1\right)\left(u_2^2 + 1\right)} \propto \int_{-X/\epsilon}^{X/\epsilon} du_1 \int_{-X/\epsilon}^{u_1} du_2 \frac{u_1 - u_2}{\left(u_1^2 + 1\right)\left(u_2^2 + 1\right)}. \tag{9}$$

This can be written as two integrals, one associated with the first term in the numerator of the integrand and the other associated with the second term. It may be seen straightforwardly that again both integrals diverge like $\log \frac{1}{\epsilon}$ as $\epsilon \to 0$.

In the general case

$$I_p(X/\epsilon) \propto \int_{-X/\epsilon}^{X/\epsilon} du_1 \int_{-X/\epsilon}^{u_1} du_2 \cdots \int_{-X/\epsilon}^{u_{p-1}} du_p \frac{\prod_{m=1}^{p-1} \prod_{n=m+1}^{p} (u_m - u_n)}{\left[\left(u_1^2 + 1\right)\left(u_2^2 + 1\right) \cdots \left(u_p^2 + 1\right)\right]^{p/2}}. \tag{10}$$

Expanding out the numerator of the integrand, I_p may be expressed as a sum of integrals, each coming from a term in the resulting series. It may be seen immediately that each integral diverges like $\log \frac{1}{\epsilon}$ as $\epsilon \to 0$. Thus when k is an integer, the GOE average of $|Z_N(\mu)|^{-k}$ diverges like

$$\epsilon^{-k(k-1)/2} \log \frac{1}{\epsilon} \tag{11}$$

as $\epsilon \to 0$.

3. GOE negative moments

Our purpose now is to prove the result obtained at the end of the previous section. We do this by making a careful asymptotic analysis of the exact GOE average defining the moments.

Regularizing the characteristic polynomial $Z_N(\mu) = \det\left(\mu \mathbf{1}_N - \hat{H}\right)$ by taking $\mathrm{Im}\,\mu > 0$, one may represent negative half-integer powers of the determinant as a Gaussian integral:

$$[Z_N(\mu)^{-n/2}] = \frac{1}{(2\pi \mathrm{i})^{nN/2}} \int \prod_{k=1}^{n} d\mathbf{S}_k \exp\left\{\frac{\mathrm{i}}{2}\mu \sum_{k=1}^{n} \mathbf{S}_k^T \mathbf{S}_k - \frac{\mathrm{i}}{2} \mathrm{Tr}\left[\hat{H} \sum_{k=1}^{n} \mathbf{S}_k \otimes \mathbf{S}_k^T\right]\right\} \tag{12}$$

where we have introduced real-valued N-dimensional vectors $\mathbf{S}_k = (s_{k,1}, \ldots, s_{k,N})^T$ for $k = 1, 2, \ldots, n$ so that $d\mathbf{S}_k = \prod_{i=1}^{N} ds_{k,i}$.

Denoting by $\langle \cdots \rangle$ the expectation value with respect to the distribution (1), our goal is to calculate the negative integer moments

$$\mathcal{K}_{N,n}^{(1)}(\mu_1) = \langle [Z_N(\mu_1)]^{-n/2} \rangle \tag{13}$$

as well as the correlation function

$$\mathcal{K}_{N,n}^{(2)}(\mu_1, \mu_2) = \langle [Z_N(\mu_1) Z_N(\mu_2^*)]^{-n/2} \rangle \tag{14}$$

assuming $\mathrm{Im}(\mu_1) = \mathrm{Im}(\mu_2) > 0$. It will be convenient for us to define $\mu_1 = \mu + \frac{\omega}{2} + \mathrm{i}\delta$ and $\mu_2^* = \mu - \frac{\omega}{2} - \mathrm{i}\delta$, with μ, ω and δ real and $\delta > 0$. Note that when $\omega = 0$, the correlation function reduces to the negative integer moments of the absolute value of the characteristic polynomial, which are the main objects of interest here.

We start with (13). Performing the ensemble averaging in the standard way using the identity

$$\int d\hat{H} \, \mathcal{P}(\hat{H}) \, e^{\pm \frac{i}{2} \text{Tr}[\hat{H}\hat{A}]} = \exp\left\{-\frac{J^2}{16N} \text{Tr}[\hat{A}^2 + \hat{A}\hat{A}^T]\right\} \tag{15}$$

gives

$$\mathcal{K}_{N,n}^{(1)}(\mu_1) = \frac{1}{(2\pi i)^{nN/2}} \int \prod_{k=1}^{n} d\mathbf{S}_k \exp\left\{\frac{i}{2}\mu_1 \sum_{k=1}^{n} \mathbf{S}_k^T \mathbf{S}_k - \frac{J^2}{8N} \sum_{k,l=1}^{n} (\mathbf{S}_k^T \mathbf{S}_l)(\mathbf{S}_l^T \mathbf{S}_k)\right\}. \tag{16}$$

Introducing an $n \times n$ real symmetric matrix $\hat{Q}$ with matrix elements $\hat{Q}_{kl} = \mathbf{S}_k^T \mathbf{S}_l$, we note that the integrand may be conveniently rewritten in the form

$$\exp\left\{\frac{i}{2}\mu_1 \text{Tr} \, \hat{Q} - \frac{J^2}{8N} \text{Tr} \, \hat{Q}^2\right\}.$$

This fact allows us to employ the 'integration theorem' proved in appendix A of [7] and to rewrite the integral in (16) in terms of an integral over the positive definite matrices $\hat{Q}$:

$$\mathcal{K}_{N,n}^{(1)} = C_{N,n}^{(1)} \int_{\hat{Q}>0} d\hat{Q} \, e^{-N[-i\mu_1 \text{Tr}\hat{Q} + \frac{1}{2}\text{Tr}\hat{Q}^2]} \det \hat{Q}^{(N-n-1)/2} \tag{17}$$

provided $N \geqslant n + 1$. We have also rescaled the integration variable $\hat{Q} \to 2N\hat{Q}$ so that the overall constant $C_{N,n}^{(1)}$ is given by

$$C_{N,n}^{(1)} = (-iN)^{Nn/2} \pi^{-\frac{n(n-1)}{4}} \frac{1}{\prod_{j=0}^{n-1} \Gamma\left(\frac{N-j}{2}\right)}$$

where $\Gamma(z)$ is the Euler gamma-function.

As the last step of the procedure we choose the eigenvalues $q_1, \ldots, q_n$ and the corresponding eigenvectors of $\hat{Q}$ as new integration variables. This corresponds to the change of the volume element

$$d\hat{Q} = \frac{1}{n!} G_n |\Delta\{\hat{q}\}| \prod_{i=1}^{n} dq_i \, d\mu(O_n) \tag{18}$$

where $\Delta\{\hat{q}\} = \prod_{i<j}(q_i - q_j)$ is the Vandermonde determinant and $d\mu(O_n)$ stands for the normalized invariant measure on the orthogonal group $O(n)$. Here

$$G_n = (\pi)^{\frac{n(n+1)}{4}} \frac{1}{\prod_{j=1}^{n} \Gamma\left(\frac{j}{2}\right)} \tag{19}$$

and the factor $1/n!$ ensures that the integration domain with respect to all variables q_k can be taken to be $0 < q_k < \infty$.

The integrand is obviously $O(n)$ invariant and so we obtain

$$\mathcal{K}_{N,n}^{(1)}(\mu_1) = \tilde{C}_{N,n}^{(1)} \int_{q_i>0} \prod_i \left(dq_i \, e^{iN(\omega/2+i\delta)q_i} q_i^{-(n+1)/2}\right) |\Delta\{\hat{q}\}| \exp\left\{-\frac{N}{2} \sum_{i=1}^{n} A(q_i)\right\} \tag{20}$$

where $\tilde{C}_{N,n}^{(1)} = \frac{1}{n!} G_n C_{N,n}^{(1)}$ and

$$A(q) = J^2 q^2 - 2i\mu q - \ln q. \tag{21}$$

We are here mainly interested in the limit of large matrix size, where one expects the results to show universality. To extract the leading asymptotics as $N \to \infty$ when n is fixed

we employ the saddle-point method, and consider $N\omega$ as well as $N\delta$ to be of the order unity when $N \to \infty$. The stationary points of $A(q_i)$ are obviously given by

$$2J^2 q_i - 2\mathrm{i}\mu - \frac{1}{q_i} = 0 \tag{22}$$

where $i = 1, 2, \ldots, n$. Each of these equations has two solutions:

$$q_{\pm} = \frac{\mathrm{i}\mu \pm \sqrt{2J^2 - \mu^2}}{2J^2}. \tag{23}$$

We would like to choose the spectral parameter μ to satisfy $|\mu| < J\sqrt{2}$ in accordance with the idea of considering the bulk of the spectrum for GOE matrices of large size. Then only for q_+ are the real parts positive, and so only in this case do the corresponding saddle points contribute to the integral over the positive semiaxis $q > 0$. Consequently, among the 2^n possible sets of saddle points $(q_{\pm}, \ldots, q_{\pm})$ only the choice

$$\hat{q}_+ = \mathrm{diag}(q_+, \ldots, q_+) \tag{24}$$

is relevant.

The presence of the Vandermonde determinants makes the integrand vanish at the saddle-point sets and so care should be taken when calculating the leading-order contribution to the integral. This turns out to be given by

$$\mathcal{K}^{(1)}_{N\gg 1,n}(\mu_1) = \tilde{C}^{(1)}{}_{N,n}(q_+)^{(N-n-1)/2}\, \mathrm{e}^{-\frac{N}{2}n[J^2 q_+^2 - 2\mathrm{i}\mu_1 q_+]} \int_{-\infty}^{\infty} \prod_{k=1}^{n} \mathrm{d}\xi_k \prod_{k_1 < k_2} \left|\xi_{k_1} - \xi_{k_2}\right| \mathrm{e}^{-\frac{t}{2}\sum_{k=1}^{n}\xi_k^2} \tag{25}$$

with

$$t = \frac{N\left(1 + 2J^2 q_+^2\right)}{2q_+^2}. \tag{26}$$

The integral in (25) is a particular case of the Selberg integral [15] and can be evaluated explicitly. We do not give the resulting expression here, because it is not needed for our purposes.

We note for later purposes that a formula for $\mathcal{K}^{(1)}_{N,n}(\mu_2^*)$ can obviously be obtained from the above expression by taking its complex conjugate and then replacing μ_1^* with μ_2^*.

We next consider the product of the expression (12) with its complex conjugate at a different value of the spectral parameter and average it over the GOE. From now on we use the index $\sigma = 1, 2$ to label the N-component vectors $\mathbf{S}_\sigma$ stemming from the first/second set of integrals. To write the resulting expression in a compact form, it is again convenient to introduce a $2n \times 2n$ matrix $\hat{Q}$ with the matrix elements $\hat{Q}^{\sigma_1,\sigma_2}_{kl} = \mathbf{S}^T_{\sigma_1,k}\mathbf{S}_{\sigma_2,l}$. Here k and l take the values $1, \ldots, n$. In terms of this matrix

$$\mathcal{K}^{(2)}_{N,n}(\mu_1, \mu_2) = \frac{1}{(2\pi)^{Nn}} \int \prod_{k=1}^{n} \mathrm{d}\mathbf{S}_{1,k}\, \mathrm{d}\mathbf{S}_{2,k} \exp\left\{ \frac{\mathrm{i}}{2}\mu_1 \sum_{k=1}^{n} \mathbf{S}^T_{1,k}\mathbf{S}_{1,k} \right.$$
$$\left. - \frac{\mathrm{i}}{2}\mu_2^* \sum_{k=1}^{n} \mathbf{S}^T_{2,k}\mathbf{S}_{2,k} - \frac{J^2}{8N}\mathrm{Tr}(\hat{Q}\hat{L}\hat{Q}\hat{L}) \right\} \tag{27}$$

where $\hat{L} = \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$. Again employing the same integration theorem as above and changing $\hat{Q} \to 2N\hat{Q}$ we arrive at

$$\mathcal{K}^{(2)}_{N,n}(\mu_1, \mu_2) = C^{(2)}_{N,n} \int_{\hat{Q}>0} \mathrm{d}\hat{Q}\, \mathrm{e}^{-\frac{N}{2}[-2\mathrm{i}\mathrm{Tr}\hat{M}\hat{Q} + J^2\mathrm{Tr}(\hat{Q}\hat{L}\hat{Q}\hat{L})]} \det\hat{Q}^{(N-2n-1)/2} \tag{28}$$

provided $N \geqslant 2n + 1$, where $\hat{M} = \text{diag}(\mu_1 \mathbf{1}_n, -\mu_2^* \mathbf{1}_n)$ and

$$C_{N,n}^{(2)} = (N)^{Nn} (\pi)^{-n(2n-1)/2} \frac{1}{\prod_{j=0}^{2n-1} \Gamma\left(\frac{N-j}{2}\right)}.$$

This equation differs from its analogue (17) in one important aspect: it is now of little use to introduce the eigenvalues/eigenvectors of $\hat{Q}$ as integration variables. Rather, it is natural to treat $\hat{Q}_L = \hat{Q}\hat{L}$ as a new matrix to integrate over. Such (non-symmetric!) matrices satisfy $\hat{Q}_L^T = \hat{L}\hat{Q}_L\hat{L}$, have all eigenvalues real and can be diagonalized by a (pseudo-orthogonal) similarity transformation $\hat{Q}_L = \hat{T}_0 \hat{q} \hat{T}_0^{-1}$, where $\hat{q} = \text{diag}(\hat{q}_1, -\hat{q}_2)$ and the $n \times n$ diagonal matrices $\hat{q}_1, \hat{q}_2$ satisfy $\hat{q}_1 > 0, \hat{q}_2 > 0$. Pseudo-orthogonal matrices $\hat{T}_0$ satisfy $\hat{T}_0^T \hat{L}\hat{T}_0 = \hat{L}$ and form the group $O(n,n)$ (the corresponding symmetry is conventionally called a 'hyperbolic symmetry' in the random matrix literature, see [17]).

It turns out that a more convenient way to proceed is to block diagonalize the matrices $\hat{Q}_L$:

$$\hat{Q}_L = \hat{T}^{-1} \begin{pmatrix} \hat{P}_1 & \\ & -\hat{P}_2 \end{pmatrix} \hat{T} \qquad \text{where} \quad \hat{T} \in \frac{O(n,n)}{O(n) \times O(n)}$$

and $\hat{P}_{1,2}$ are $n \times n$ real symmetric, with positive eigenvalues $\hat{q}_{1,2}$, respectively. The integration measure $[\mathrm{d}\hat{Q}_L]$ can be derived in terms of the new variables following the standard steps (see, e.g., [20]) outlined in the appendix of the present paper. We arrive at $[\mathrm{d}\hat{Q}] = A\mathrm{d}\hat{P}_1 \mathrm{d}\hat{P}_2 \prod_{k_1,k_2} (q_{1,k_1} + q_{2,k_2}) \mathrm{d}\mu(T)$, where $A = G_n^2/[n! 2^{n(n+1)/2}]$ and the last factor is the invariant measure on the manifold of T-matrices. An explicit expression for this is presented, for reference purposes, in the appendix.

After all these preparatory steps we arrive at the following expression:

$$\begin{aligned}
\mathcal{K}_{N,n}^{(2)} = A C_{N,n}^{(2)} \int_{\hat{P}_1 > 0} \int_{\hat{P}_1 > 0} \mathrm{d}\hat{P}_1 \, \mathrm{d}\hat{P}_2 \, I(\hat{M}, \hat{P}_1, \hat{P}_2) \\
\times \prod_{k_1,k_2} (q_{1,k_1} + q_{2,k_2}) \det[-\hat{P}_1 \hat{P}_2]^{(N-2n-1)/2} \, \mathrm{e}^{-\frac{NJ^2}{2} \text{Tr}(\hat{P}_1^2 + \hat{P}_2^2)}
\end{aligned} \tag{29}$$

where

$$I(\hat{M}, \hat{P}_1, \hat{P}_2) = \int \mathrm{d}\mu(\hat{T}) \exp\left\{ \mathrm{i}N \, \text{Tr} \begin{pmatrix} \hat{\mu}_1 \mathbf{1}_n & \\ & \mu_2^* \mathbf{1}_n \end{pmatrix} \hat{T}^{-1} \begin{pmatrix} \hat{P}_1 & \\ & -\hat{P}_2 \end{pmatrix} \hat{T} \right\}. \tag{30}$$

Employing the explicit parametrization for the matrices T given in the appendix we can rewrite the above integral as

$$I(\hat{M}, \hat{P}_1, \hat{P}_2) = \mathrm{e}^{\mathrm{i}N \frac{\mu_1 + \mu_2^*}{2} \sum_k (q_{1k} - q_{2k})} I_0(\hat{M}, \hat{P}_1, \hat{P}_2) \tag{31}$$

where

$$\begin{aligned}
I_0(\hat{M}, \hat{P}_1, \hat{P}_2) = \int_{-\infty}^{\infty} \prod_{k=1}^{n} \mathrm{d}\psi_k \prod_{k_1 < k_2} |\cosh\psi_{k_2} - \cosh\psi_{k_2}| \int [\mathrm{d}\mu(O_L)][\mathrm{d}\mu(O_R)] \\
\times \exp\left\{ \mathrm{i}N \frac{\mu_1 - \mu_2^*}{2} \text{Tr} \cosh\hat{\psi} [\hat{O}_L^T \hat{P}_1 \hat{O}_L + \hat{O}_R^T \hat{P}_2 \hat{O}_R] \right\}
\end{aligned} \tag{32}$$

$\hat{O}_{L,R} \in O(n)$, and $\hat{\psi}$ is diagonal.

In the case of GUE matrices studied in [6] a helpful trick under similar conditions was to perform the (unitary) group integrals explicitly by employing the famous Itzykson–Zuber–Harish-Chandra integration formula [10]. The lack of an analogous formula for the orthogonal group forces us to take a slightly different route.

It is easy to see that the value of this integral can depend only on the eigenvalue matrices $\hat{q}_1$ and $\hat{q}_2$. Let us therefore introduce the eigenvalues (and corresponding eigenvectors) of the Hermitian matrices $\hat{P}_1 > 0$ and $\hat{P}_2 > 0$ as the integration variables. This results in the following expression:

$$
\mathcal{K}_{N,n}^{(2)}(\mu_1, \mu_2) = \tilde{C}_{N,n}^{(2)} \int_0^\infty \prod_i dq_{1,i} q_{1,i}^{-n-1/2} |\Delta\{\hat{q}_1\}| \int_0^\infty \prod_i dq_{2,i} q_{2,i}^{-n-1/2} |\Delta\{\hat{q}_1\}|
$$
$$
\times \prod_{k_1,k_2} \left(q_{1,k_1} + q_{2,k_2}\right) I(\hat{M}, \hat{q}_1, \hat{q}_2) \exp\left(-\frac{N}{2}\sum_{i=1}^n A(q_{1,i}) - \frac{N}{2}\sum_{i=1}^n A^*(q_{2,i})\right)
$$
$$
\tag{33}
$$

where

$$
\tilde{C}_{N,n}^{(2)} = \frac{1}{2^{n(n+1)/2} n!^3} G_n^4 C_{N,n}^{(2)}
\tag{34}
$$

$$
A(q) = J^2 q^2 - 2i\mu q - \ln q \qquad \text{and} \qquad A^*(q) = J^2 q^2 + 2i\mu q - \ln q.
\tag{35}
$$

Again, we need to perform an asymptotic analysis as $N \to \infty$. The most interesting regime occurs when one keeps the difference $\mathrm{Re}(\mu_1 - \mu_2^*) \equiv \omega$ and the regularization δ so small as to ensure $N \max(\omega, \delta) < \infty$, while $\mu = \mathrm{Re}\,\frac{(\mu_1+\mu_2)}{2}$ is kept in the range $|\mu| < J\sqrt{2}$.

The stationary points of $A(q)$ and $A^*(q)$ are now given by

$$
q_{1,i} - i\mu - \frac{1}{q_{1,i}} = 0 \qquad \text{and} \qquad q_{2,i} + i\mu - \frac{1}{q_{2,i}} = 0
\tag{36}
$$

where $i = 1, 2, \ldots, n$. Each of these two equations has two solutions:

$$
q_{1\pm} = \frac{i\mu \pm \sqrt{2J^2 - \mu^2}}{2J^2} \qquad \text{and} \qquad q_{2\pm} = \frac{-i\mu \pm \sqrt{2J^2 - \mu^2}}{2J^2}
\tag{37}
$$

but only for $q_{1+}, q_{2+} = q_{1+}^*$ are the real parts positive; that is, only then do the corresponding saddle points contribute to the integral over the positive semiaxis $q_{1,i} > 0$ or $q_{2,i} > 0$. Consequently, among the 2^{2n} possible sets of stationary points, only the choice

$$
\hat{q}_1 = \mathrm{diag}(q_{1+}, \ldots, q_{1+}) \qquad \hat{q}_1 = \mathrm{diag}(q_{1+}^*, \ldots, q_{1+}^*)
\tag{38}
$$

is relevant. This is a major simplification, because for such a choice the integrand in (31) turns out to be independent of the matrices $\hat{O}_1$, $\hat{O}_2$.

Taking care of the Vandermonde determinants when calculating the fluctuations around the chosen saddle points and remembering that

$$
q_1 + q_1^* = \pi\rho(\mu) \qquad q_1 q_1^* = 1/2J^2
\tag{39}
$$

where $\rho(\mu) = \frac{1}{\pi J^2}\sqrt{2J^2 - \mu^2}$ is the mean density of eigenvalues for GOE matrices, we observe that when the asymptotic expression for the correlation function under consideration is divided by the product of the negative moments (25) the Selberg integrals cancel out, as well as all the exponential factors too. The resulting expression amounts to

$$
\mathcal{K}_n(\mu_1, \mu_2) = \lim_{N \to \infty} \frac{\langle [\det(\mu_1 \mathbf{1}_N - \hat{H})\det(\mu_2^* \mathbf{1}_N - \hat{H})]^{-n/2} \rangle}{\langle \det(\mu_1 \mathbf{1}_N - \hat{H})^{-n/2} \rangle \langle \det(\mu_2^* \mathbf{1}_N - \hat{H})^{-n/2} \rangle} = \mathcal{C} \times F_n^{\mathrm{GOE}}(\epsilon)
\tag{40}
$$

where

$$
F_n^{\mathrm{GOE}}(\epsilon) = e^{n\epsilon} \int_1^\infty \prod_{k=1}^n \frac{d\lambda_k}{\sqrt{\lambda_k^2 - 1}} \prod_{k_1 < k_2} |\lambda_{k_1} - \lambda_{k_2}| e^{-\epsilon \sum_{k=1}^n \lambda_k}
\tag{41}
$$

in which we have introduced variables $\lambda_k = \cosh \psi_k \in [1, \infty)$,

$$\epsilon = -iN\pi\rho(\mu)(\mu_1 - \mu_2^*)/2 \tag{42}$$

and

$$\mathcal{C} = (\pi\rho J)^{n^2} \left(\frac{N}{2}\right)^{n^2/2} \frac{(2\pi)^{n/2}}{n!} \frac{1}{\left[\prod_{j=1}^{n} \Gamma\left(\frac{j}{2}\right)\right]^2}. \tag{43}$$

This expression is valid for all $|\epsilon| < \infty$, i.e. as far as $(\mu_1 - \mu_2^*) = O(1/N)$, and constitutes one of the main results of the present paper. In the subsequent analysis we concentrate on the moments of characteristic polynomials and thus treat ϵ as a real parameter.

It is instructive to compare (41) with its counterpart for the Gaussian unitary ensemble (see [21]; in [6] the corresponding expression is implicit):

$$F_n^{\text{GUE}}(\epsilon) = e^{n\epsilon} \int_1^\infty \prod_{k=1}^{n} d\lambda_k \prod_{k_1 < k_2} \left(\lambda_{k_1} - \lambda_{k_2}\right)^2 e^{-\epsilon \sum_{k=1}^{n} \lambda_k}. \tag{44}$$

The latter integral is a specific case of the Selberg integral [15] and can be immediately evaluated, yielding

$$F_n^{\text{GUE}}(\epsilon) = \frac{1}{\epsilon^{n^2}} \prod_{j=0}^{n-1} j!(j+1)!. \tag{45}$$

Such a formula exemplifies a 'normal' dependence of the negative moments on ϵ: namely, one can extract the rate of divergence as $\epsilon \to 0$ by analysing the perturbative expansion of the integral as $\epsilon \to \infty$. Performing the latter limit is effectively the same as considering the case when $\delta = \text{Im}\,\mu_{1,2}$ is left unscaled by the mean eigenvalue density (see the introduction). In other words, it is equivalent to considering the limit $\delta \to 0$ *after* taking $N \to \infty$. Thus for the GUE the asymptotics of the negative moments is the same irrespective of the order in which limits are taken, and so is relatively uninteresting.

The integral (41) behaves in this sense 'anomalously'. It does not belong to the class of Selberg integrals and apart from when $n = 1$ (in which case it just yields the Macdonald function $K_0(\epsilon)$) we have failed to evaluate it explicitly in a simple closed form. We therefore proceed to analyse the limits $\epsilon \to \infty$ and $\epsilon \to 0$ separately.

In the perturbative region $\epsilon \gg 1$ the integral is obviously dominated by a small vicinity of the lower limit: $\lambda_k - 1 \ll 1$. Introducing variables $x_k \in [0, \infty)$ such that $\lambda_k = 1 + x_k/\epsilon$, we immediately see that asymptotically the integral is again of Selberg type:

$$F_n^{\text{GOE}}(\epsilon \gg 1) = \frac{1}{2^{n/2}\epsilon^{n^2/2}} \int_0^\infty \prod_{k=1}^{n} \frac{dx_k}{\sqrt{x_k}} \prod_{k_1 < k_2} |x_{k_1} - x_{k_2}| e^{-\sum_{k=1}^{n} x_k} \tag{46}$$

$$= \frac{1}{2^{n/2}\epsilon^{n^2/2}} \prod_{j=0}^{n-1} \frac{\Gamma\left(\frac{3+j}{2}\right)\Gamma\left(\frac{1+j}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{\epsilon^{n^2/2}} \frac{n!}{(2\pi)^{n/2}} \left[\prod_{j=1}^{n} \Gamma\left(\frac{j}{2}\right)\right]^2. \tag{47}$$

We then see that the perturbative behaviour for GOE moments is essentially of the same type as that for GUE moments:

$$\mathcal{K}_n(\mu_1, \mu_2) = (\pi\rho(\mu)J)^{n^2} \left(\frac{N}{2\epsilon}\right)^{n^2/2} = \left(\frac{\pi\rho(\mu)J}{-i[\mu_1 - \mu_2^*]/J}\right)^{n^2/2}. \tag{48}$$

In contrast to this, in the non-perturbative region $\epsilon \to 0$ the behaviour of the GUE and GOE moments is very different. In this limit the integral is dominated by $\lambda_k \sim \epsilon^{-1} \gg 1$ and it is natural to introduce rescaled variables $y_k = \epsilon \lambda_k$, leading to

$$F_n^{\text{GOE}}(\epsilon \ll 1) = \frac{1}{\epsilon^{n(n-1)/2}} \int_\epsilon^\infty \cdots \int_\epsilon^\infty \prod_{k=1}^n \frac{\mathrm{d} y_k}{y_k} \prod_{k_1 < k_2} |y_{k_1} - y_{k_2}| \, \mathrm{e}^{-\sum_{k=1}^n y_k}. \tag{49}$$

Note that one cannot set the lower limit of integration with respect to the variables y_k to be zero, because the corresponding integrals diverge logarithmically there. To extract the leading-order behaviour as $\epsilon \to 0$ we differentiate the function $\tilde{F}_n(\epsilon) = \epsilon^{n(n-1)/2} F_n^{\text{GOE}}(\epsilon \ll 1)$ with respect to its argument, reducing it asymptotically to a Selberg-type integral

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \tilde{F}_n(\epsilon) = -\frac{n}{\epsilon} \, \mathrm{e}^{-\epsilon} \int_\epsilon^\infty \frac{\mathrm{d} y_2}{y_2} \, \mathrm{e}^{-y_2} \cdots \int_\epsilon^\infty \frac{\mathrm{d} y_n}{y_n} \, \mathrm{e}^{-y_n} (y_2 - \epsilon) \cdots (y_n - \epsilon) \prod_{2 \leqslant k < l \leqslant n} |y_k - y_l| \tag{50}$$

$$\longrightarrow -\frac{n}{\epsilon} \int_0^\infty \prod_{k=1}^{n-1} \mathrm{d} y_k \, \mathrm{e}^{-y_k} \prod_{1 \leqslant k < l \leqslant (n-1)} |y_k - y_l| = -\frac{n}{\epsilon} \prod_{j=0}^{n-2} \frac{\Gamma\left(\frac{3+j}{2}\right) \Gamma\left(\frac{2+j}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}. \tag{51}$$

Thus, we conclude that for all integers $n \geqslant 1$

$$F_n^{\text{GOE}}(\epsilon \to 0) = \frac{2^{n-1}}{\pi^{n/2}} n \prod_{j=0}^{n-1} \left[\Gamma\left(1 + \frac{j}{2}\right) \right] \left[\Gamma\left(\frac{j+1}{2}\right) \right] \frac{\ln 1/\epsilon}{\epsilon^{n(n-1)/2}}. \tag{52}$$

This constitutes another of our main results, the point being that it agrees with the asymptotic expression (11) predicted by the heuristic theory of dominating singularities outlined in section 2.

Acknowledgments

Appendix. Calculation of the Jacobian

To evaluate the integral in (28) one needs to calculate the Jacobian generated by the variable transformation $\hat{Q}_L = \hat{T}^{-1} \hat{P} \hat{T}$, with $\hat{P} = \text{diag}(\hat{P}_1, -\hat{P}_2) = \hat{P}^T$ and $\hat{P}_1 > 0$, $\hat{P}_2 > 0$ being real symmetric $n \times n$ matrices. For the matrices $\hat{T} \in \frac{O(n,n)}{O(n) \times O(n)}$ we employ the following explicit parametrization in terms of a real $n \times n$ matrix $\hat{t}$:

$$\hat{T} = \begin{pmatrix} \sqrt{1 + \hat{t}\hat{t}^T} & \hat{t} \\ \hat{t}^T & \sqrt{1 + \hat{t}^T \hat{t}} \end{pmatrix} \qquad \text{hence} \qquad \hat{T}^{-1} = \begin{pmatrix} \sqrt{1 + \hat{t}\hat{t}^T} & -\hat{t} \\ -\hat{t}^T & \sqrt{1 + \hat{t}^T \hat{t}} \end{pmatrix}.$$

It is convenient to follow the scheme suggested in [20]. One starts by considering the relation between the matrix differentials:

$$\mathrm{d}\hat{Q} = \hat{T}^{-1} \, \mathrm{d}\hat{\tilde{Q}} \hat{T} \qquad \mathrm{d}\hat{\tilde{Q}} = \mathrm{d}\hat{P} + (\hat{P} \, \mathrm{d}\hat{\tau} - \mathrm{d}\hat{\tau} \hat{P}) \tag{53}$$

where we have introduced the notation $\mathrm{d}\hat{\tau} = \mathrm{d}\hat{T} \tilde{T}^{-1}$. To calculate the Jacobian the difference between $\mathrm{d}\hat{Q}$ and $\mathrm{d}\hat{\tilde{Q}}$ is immaterial and we omit the tilde henceforth. Partitioning the matrix $\mathrm{d}\hat{Q}$ into four $n \times n$ sub-blocks $\mathrm{d}\hat{q}_{pq}$, $p, q = 1, 2$, one then rewrites the above relation blockwise:

$$\begin{aligned} \mathrm{d}\hat{q}_{11} &= \mathrm{d}\hat{P}_1 + (\hat{P}_1 \mathrm{d}\hat{\tau}_{11} - \mathrm{d}\hat{\tau}_{11} \hat{P}_1) & \mathrm{d}\hat{q}_{22} &= -\mathrm{d}\hat{P}_2 - (\hat{P}_2 \, \mathrm{d}\hat{\tau}_{22} - \mathrm{d}\hat{\tau}_{22} \hat{P}_2) \\ \mathrm{d}\hat{q}_{12} &= \hat{P}_1 \, \mathrm{d}\hat{\tau}_{12} + \mathrm{d}\hat{\tau}_{12} \hat{P}_2 & \mathrm{d}\hat{q}_{21} &= \mathrm{d}\hat{q}_{12}^T. \end{aligned}$$

Inspecting the block structure of the corresponding Jacobian, symbolically written as $J = \det(\mathrm{d}[\hat{q}_{11}, \hat{q}_{22}, \hat{q}_{12}]/\mathrm{d}[\hat{P}_1, \hat{P}_2, \hat{\tau}])$, one may easily verify that

$$J = \det(\mathrm{d}[\hat{q}_{12}]/\mathrm{d}[\hat{\tau}_{12}]) = \det(\hat{P}_1 \otimes \mathbf{1}_n + \mathbf{1}_n \otimes \hat{P}_2) = \prod_{i,j}^{n}(q_{1,i} + q_{2,j})$$

where $q_{1,i}$ and $q_{2,i}$ are (positive) eigenvalues of the matrices $\hat{P}_1$, $\hat{P}_2$. Then an intermediate result for the measure can be schematically written as

$$\mathrm{d}\hat{Q} = \prod_{i,j}^{n}(q_{1,i} + q_{2,j})\,\mathrm{d}\hat{P}_1\,\mathrm{d}\hat{P}_2\,\det(\mathrm{d}[\hat{\tau}_{12}]/\mathrm{d}[\hat{t}])\,\mathrm{d}\hat{t}$$

where, explicitly

$$-\mathrm{d}[\hat{\tau}_{12}] = [\mathrm{d}T_{11}][T^{-1}]_{12} + [\mathrm{d}T]_{12}[T^{-1}]_{22} = \mathrm{d}\big[\sqrt{1 + \hat{t}\hat{t}^T}\big]\hat{t} + \mathrm{d}\hat{t}\sqrt{1 + \hat{t}^T\hat{t}}.$$

To calculate the remaining determinant we employ the singular value decomposition $\hat{t} = \hat{O}_L^{-1}\sinh\hat{\theta}\,\hat{O}_R$ expressing $\hat{t}$ in terms of the two real orthogonal $n \times n$ matrices $\hat{O}_{L,R} \in O(n)$ and a real diagonal matrix $\hat{\theta} = \mathrm{diag}\,(\theta_1, \ldots, \theta_n)$, assuming, for uniqueness, $\theta_1 > \cdots > \theta_n$. Then $\sqrt{1 + \hat{t}\hat{t}^T} = \hat{O}_L^{-1}\cosh\hat{\theta}\,\hat{O}_L$ and $\sqrt{1 + \hat{t}^T\hat{t}} = \hat{O}_R^{-1}\cosh\hat{\theta}\,\hat{O}_R$. Further introducing $\mathrm{d}\hat{v}_{L,R} = \mathrm{d}\hat{O}_{L,R}[\hat{O}_{L,R}]^{-1}$ and $\mathrm{d}\hat{\tilde{\tau}} = \hat{O}_L\,\mathrm{d}[\hat{\tau}_{12}]\hat{O}_R^{-1}$ we find, after straightforward manipulations

$$\mathrm{d}\hat{\tilde{\tau}} = \mathrm{d}\hat{\theta} + \sinh\hat{\theta}\,\mathrm{d}\hat{v}_R\cosh\hat{\theta} - \cosh\hat{\theta}\,\mathrm{d}\hat{v}_L\sinh\hat{\theta}.$$

Next, differentiating $\hat{O}_{L,R}\hat{O}_{L,R}^T = 1$, we observe that $\mathrm{d}\hat{v}_{L,R}$ must be antisymmetric, hence $\mathrm{d}\hat{v}_{ii} = 0$ and $\mathrm{d}\hat{v}_{j\neq i} = -\mathrm{d}\hat{v}_{ij}$, from which it is clear that $[\mathrm{d}\hat{\tilde{\tau}}]_{ii} = \theta_i$ for all $i = 1, \ldots, n$. At the same time, for any of the $n(n-1)/2$ pairs $1 \leqslant i < j \leqslant n$ we have, in vector notation, the relation between the differentials

$$\begin{pmatrix}\mathrm{d}\tilde{\tau}_{ij} \\ \mathrm{d}\tilde{\tau}_{ji}\end{pmatrix} = \begin{pmatrix}\sinh\theta_i\cosh\theta_j & -\cosh\theta_i\sinh\theta_j \\ -\cosh\theta_i\sinh\theta_j & \sinh\theta_i\cosh\theta_j\end{pmatrix}\begin{pmatrix}(\mathrm{d}\hat{v}_R)_{ij} \\ (\mathrm{d}\hat{v}_L)_{ij}\end{pmatrix}. \tag{54}$$

The Jacobian in question then reduces to a product of the determinants of the matrices entering in the above equation, which are simply $|\sinh^2\theta_i - \sinh^2\theta_j| = |\cosh 2\theta_i - \cosh 2\theta_j|/2$. Finally, we introduce $\psi_i = 2\theta_i$, remove the relative ordering of ψ_i in favour of the factor $1/n!$ in the measure and remember that $[\mathrm{d}\hat{v}_{L,R}]$ gives rise to the product of invariant measures $[\mathrm{d}\mu(O_{L,R})]$ on the orthogonal group $O(n)$ (which we assumed to be normalized to unity). The measure $[\mathrm{d}\hat{Q}_L]$ in the coordinates $\hat{P}_{1,2}, \hat{\psi}, \hat{O}_{L,R}$ then assumes the following form:

$$[\mathrm{d}\hat{Q}] = \frac{G_n^2}{2^{n(n+1)/2}n!}\prod_{i,j}^{n}(q_{1,i} + q_{2,j})$$

$$\times \prod_{1\leqslant i<j\leqslant n}|\cosh\psi_i - \cosh\psi_j|\,\mathrm{d}\hat{P}_1\,\mathrm{d}\hat{P}_2\,\mathrm{d}\hat{\psi}\,[\mathrm{d}\mu(O_L)][\mathrm{d}\mu(O_R)] \tag{55}$$

where $-\infty \leqslant \psi_i < \infty$ for $i = 1, \ldots, n$.

To conclude, we give the explicit expression for the following combination used in the main text:

$$\mathrm{Tr}\left[\begin{pmatrix}\hat{\mu}_1\mathbf{1}_n & \\ & \mu_2^*\mathbf{1}_n\end{pmatrix}\hat{T}^{-1}\begin{pmatrix}\hat{P}_1 & \\ & -\hat{P}_2\end{pmatrix}\hat{T}\right] = \mathrm{Tr}\begin{pmatrix}\hat{\mu}_1\mathbf{1}_n & \\ & \mu_2^*\mathbf{1}_n\end{pmatrix}\begin{pmatrix}\cosh\hat{\psi}/2 & -\sinh\hat{\psi}/2 \\ -\sinh\hat{\psi}/2 & \cosh\hat{\psi}/2\end{pmatrix}$$

$$\times \begin{pmatrix}\hat{P}_L & \\ & -\hat{P}_R\end{pmatrix}\begin{pmatrix}\cosh\hat{\psi}/2 & \sinh\hat{\psi}/2 \\ \sinh\hat{\psi}/2 & \cosh\hat{\psi}/2\end{pmatrix}$$

$$= \mathrm{Tr}[\hat{P}_L(\mu_1\cosh^2\hat{\psi}/2 - \mu_2^*\sinh^2\hat{\psi}/2)]$$

$$- \mathrm{Tr}[\hat{P}_R(\mu^*\cosh^2\hat{\psi}/2 - \mu\sinh^2\hat{\psi}/2)]$$

$$= \tfrac{1}{2}(\mu_1 + \mu_2^*)\mathrm{Tr}(\hat{P}_L - \hat{P}_R) + \tfrac{1}{2}(\mu_1 - \mu_2^*)(\mathrm{Tr}\,\hat{P}_L\cosh\hat{\psi} + \mathrm{Tr}\,\hat{P}_R\cosh\hat{\psi}) \tag{56}$$

where we have introduced matrices $\hat{P}_L = \hat{O}_L \hat{P}_1 \hat{O}_L^{-1}$ and $\hat{P}_R = \hat{O}_R \hat{P}_2 \hat{O}_R^{-1}$ having the same eigenvalues $\hat{q}_1 = \text{diag}(q_{1,1}, \ldots, q_{1,n})$ and $\hat{q}_2 = \text{diag}(q_{2,1}, \ldots, q_{2,n})$ as the matrices $\hat{P}_{1,2}$.

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